Lezioni di Ricerca Operativa

Corso di Laurea in Informatica ed Informatica Applicata

Università di Salerno

Lezione n° 8: Esercitazione

- Variazione del gradiente
- Introduzione di vincoli nel problema
- Calcolo delle direzioni estreme

Prof. Cerulli – Dott.ssa Gentili – Dott. Carrabs

Dato il seguente problema di programmazione lineare:

$$\min \; z = 3x_1 + x_2$$

$$
\begin{array}{lll}
(1) & \frac{1}{2}x_1 - x_2 & \leq 3 \\
(2) & -x_1 + x_2 & \leq 1 \\
(3) & 2x_1 - x_2 & \geq 2 \\
(4) & x_1, x_2 & \geq 0
\end{array}
$$

a) Risolvere graficamente il problema

$$\min \quad z = 3x_1 + x_2$$

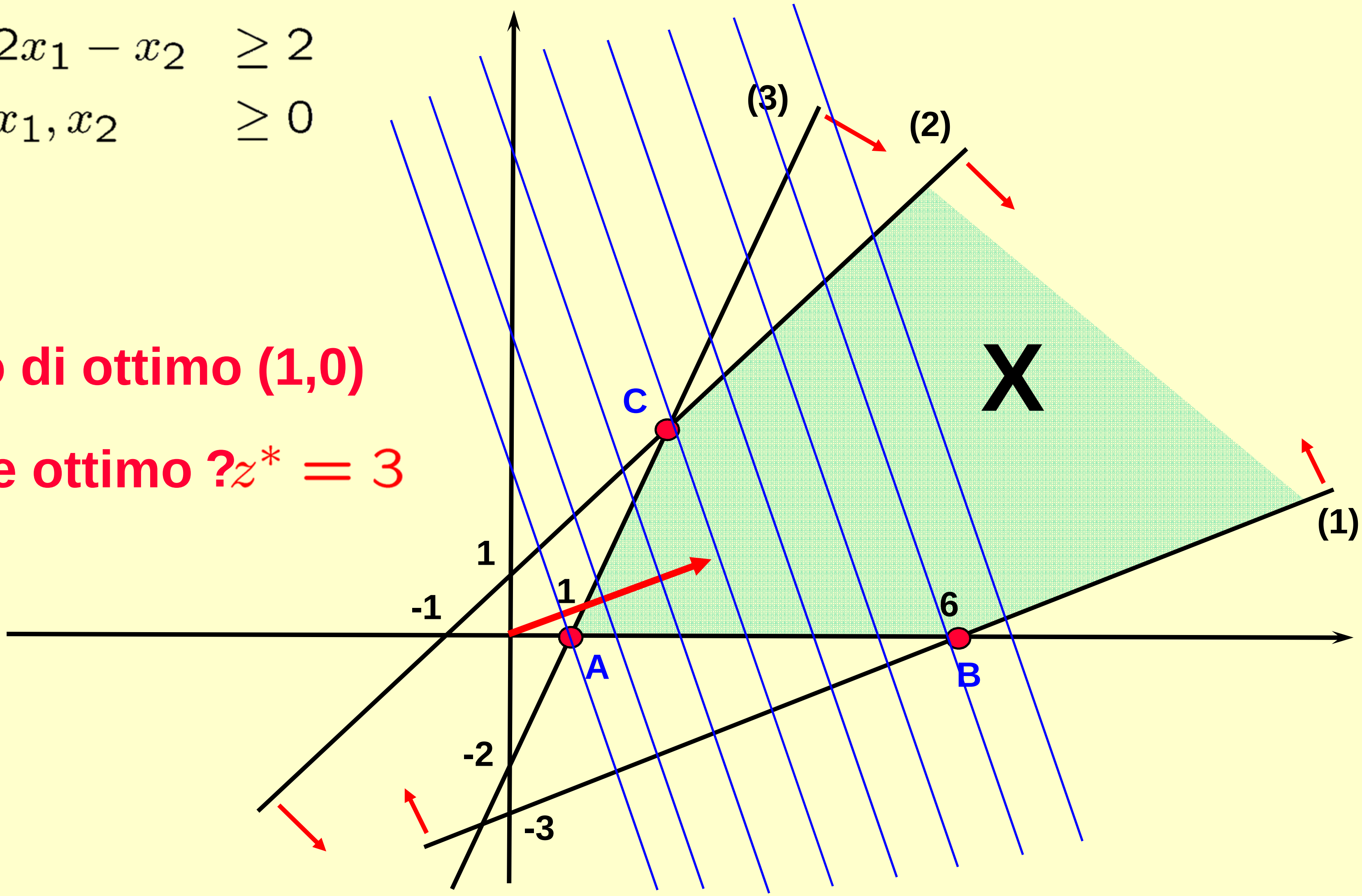
$$(1) \quad \frac{1}{2}x_1 - x_2 \leq 3$$

$$(2) \quad -x_1 + x_2 \leq 1$$

$$(3) \quad 2x_1 - x_2 \geq 2$$

$$(4) \quad x_1, x_2 \geq 0$$

Gradiente (3,1)



Punto di ottimo (1,0)

Valore ottimo ? $z^* = 3$

$$\min \quad z = -x_1 - x_2$$

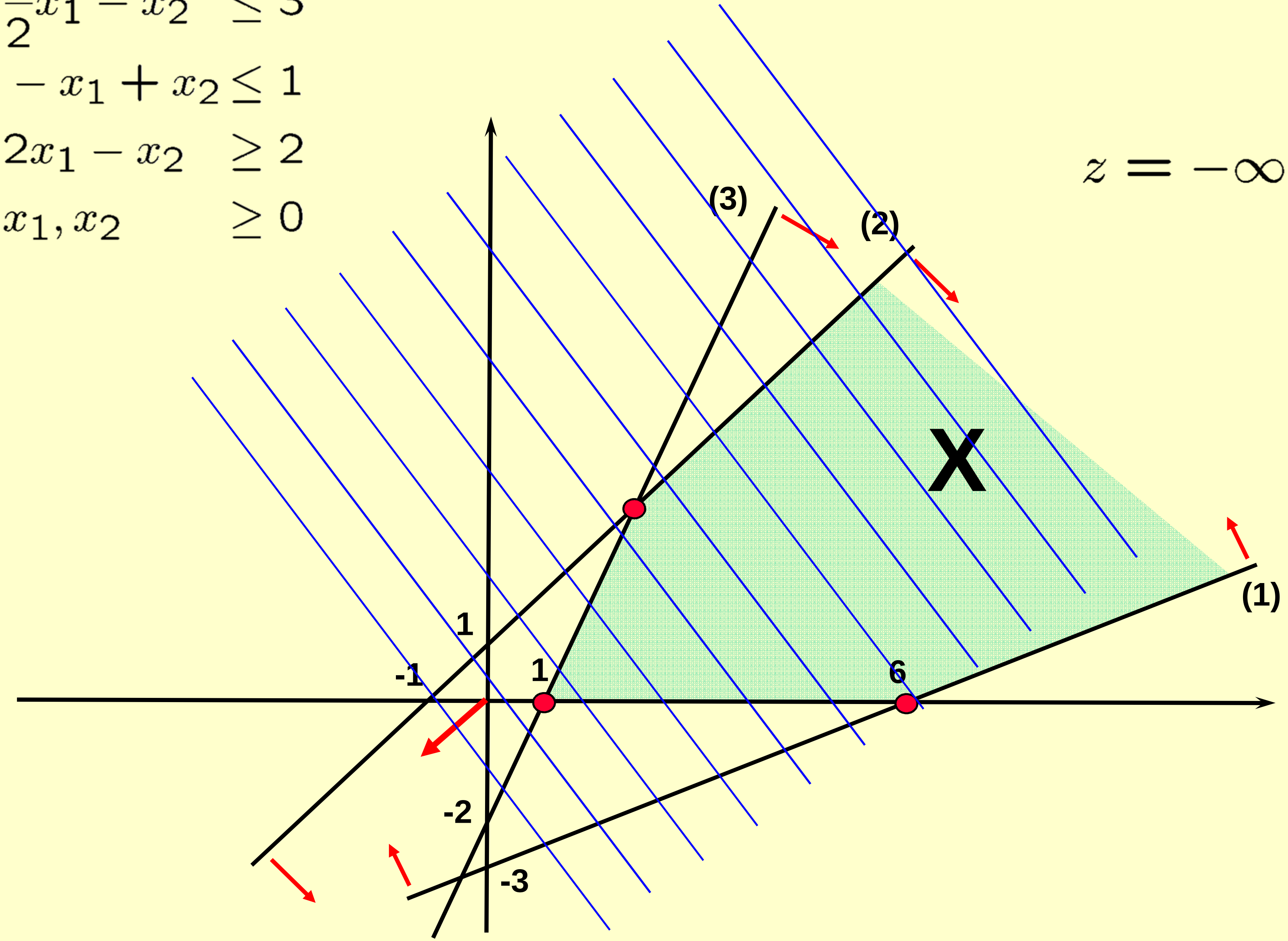
$$(1) \quad \frac{1}{2}x_1 - x_2 \leq 3$$

$$(2) \quad -x_1 + x_2 \leq 1$$

$$(3) \quad 2x_1 - x_2 \geq 2$$

$$(4) \quad x_1, x_2 \geq 0$$

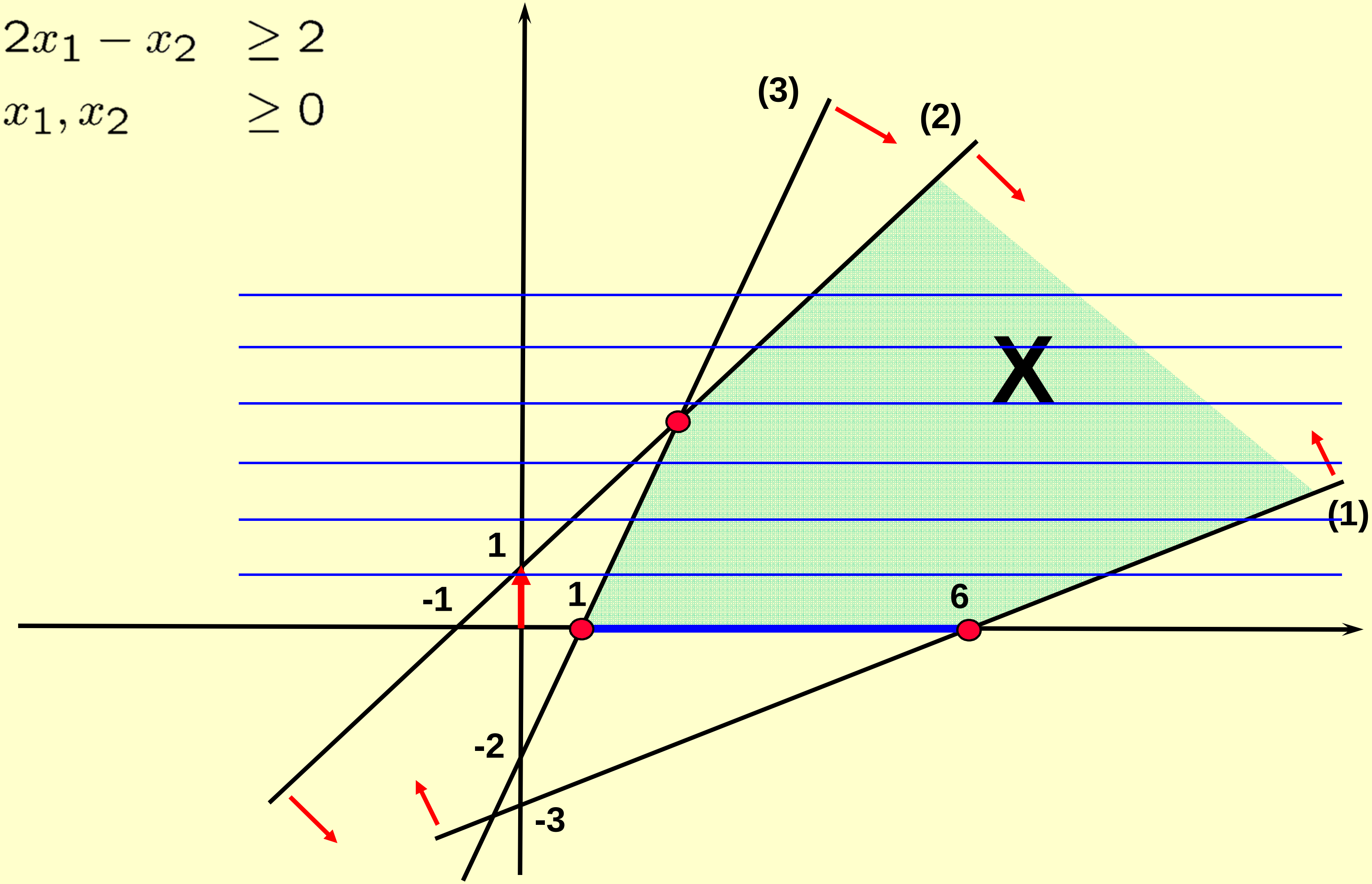
b) Determinare una nuova funzione obiettivo che abbia ottimo illimitato



c) Determinare una nuova funzione obiettivo che abbia infiniti punti di ottimo

$$\min \quad z = x_2$$

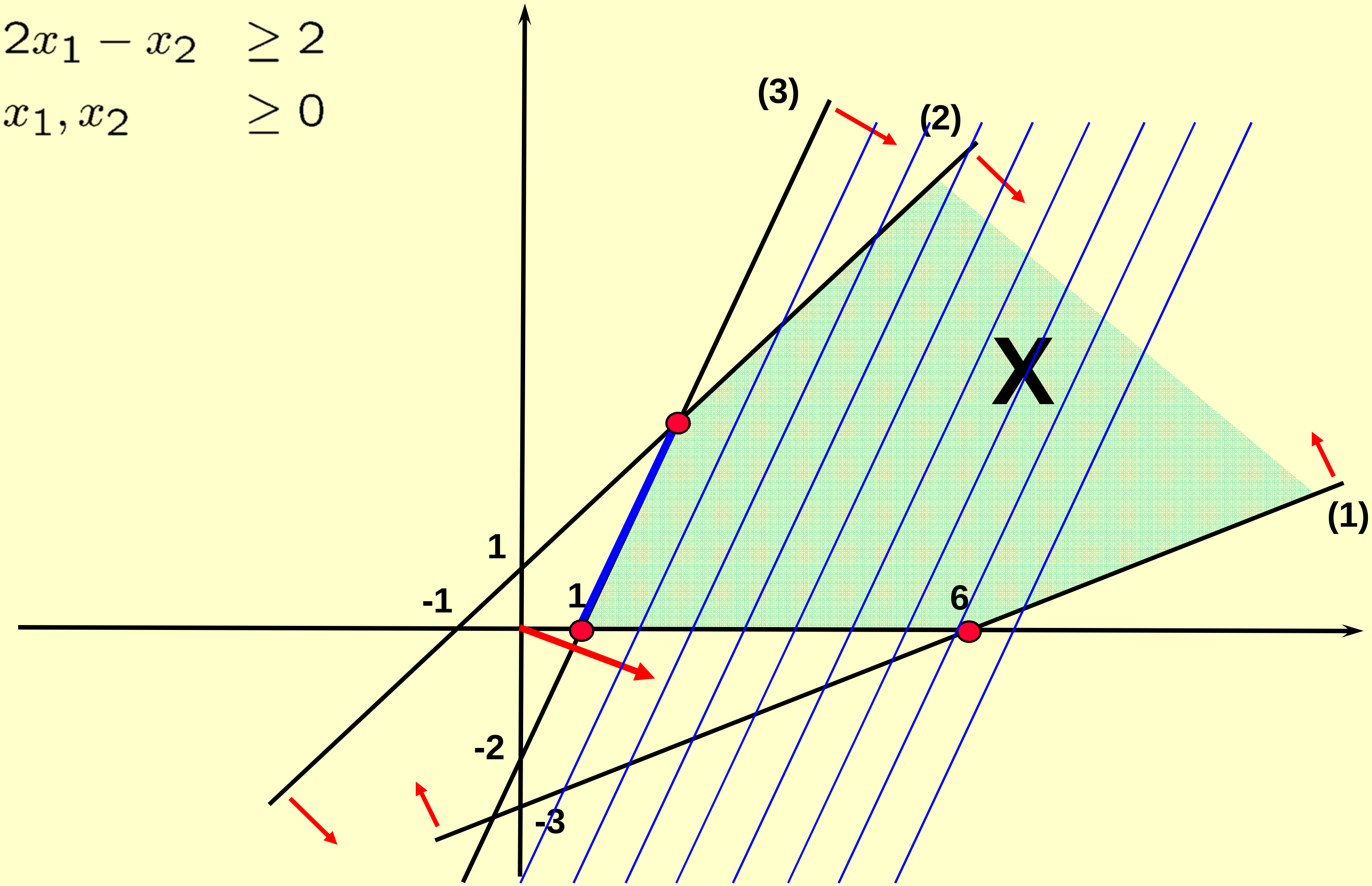
$$
\begin{array}{lll}
(1) & \frac{1}{2}x_1 - x_2 & \leq 3 \\
(2) & -x_1 + x_2 & \leq 1 \\
(3) & 2x_1 - x_2 & \geq 2 \\
(4) & x_1, x_2 & \geq 0
\end{array}
$$



$$\min \quad z = 2x_1 - x_2$$

$$\begin{array}{lll} (1) & \frac{1}{2}x_1 - x_2 & \leq 3 \\ (2) & -x_1 + x_2 & \leq 1 \\ (3) & 2x_1 - x_2 & \geq 2 \\ (4) & x_1, x_2 & \geq 0 \end{array}$$

c) Determinare una nuova funzione obiettivo che abbia infiniti punti di ottimo



$$max\ z = -3x_1 + 2x_2$$

$$(1) \quad \frac{1}{2}x_1 - x_2 \leq 3$$

$$(2) \quad -x_1 + x_2 \leq 1$$

$$(3) \quad 2x_1 - x_2 \geq 2$$

$$(4) \quad x_1, x_2 \geq 0$$

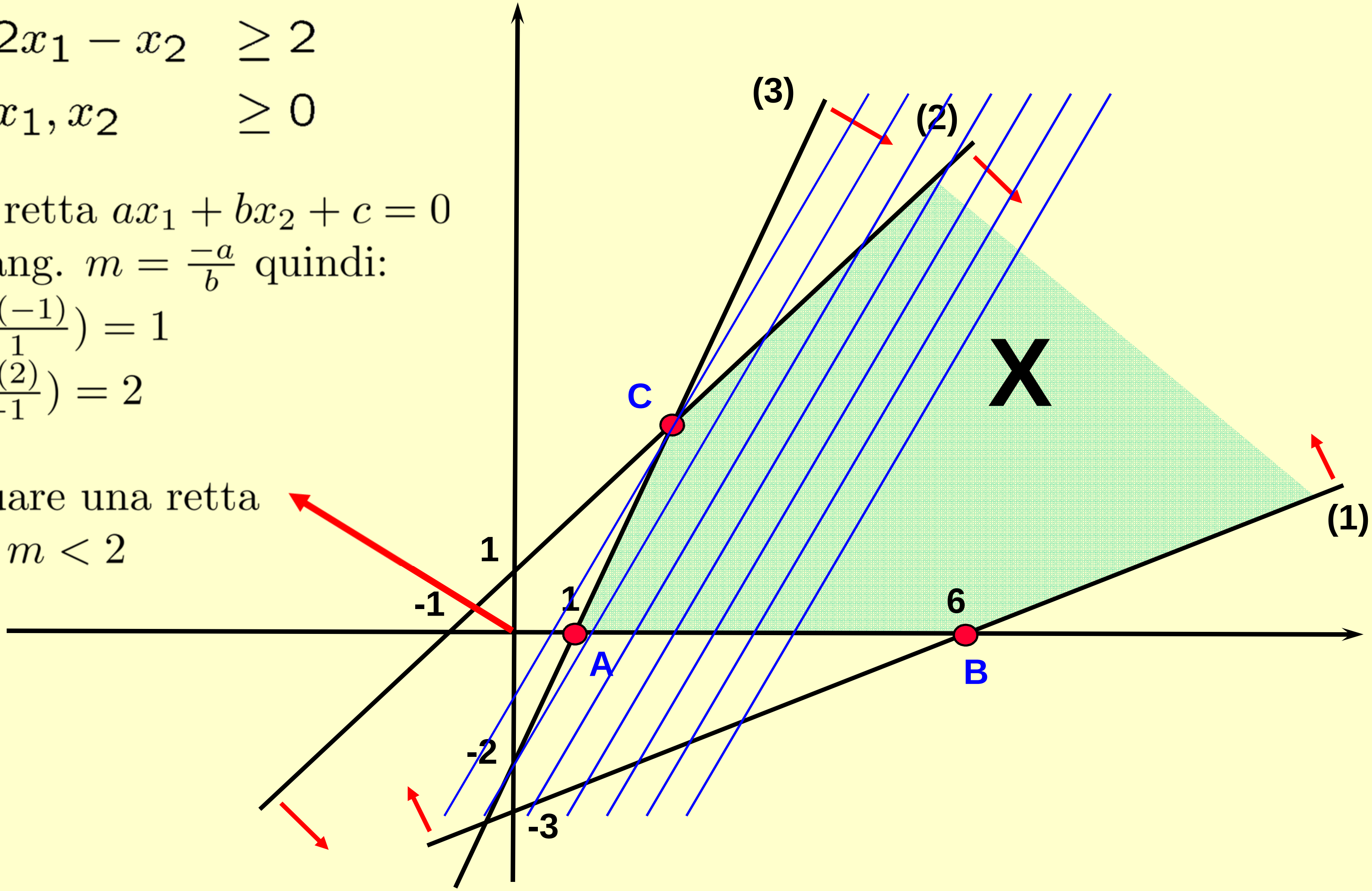
d) Determinare una nuova funzione obiettivo che renda C punto di ottimo unico

Data la retta $ax_1 + bx_2 + c = 0$ il coeff.ang. $m = \frac{-a}{b}$ quindi:

$m_2 = \frac{-(-1)}{1}) = 1$

$m_3 = \frac{-(2)}{-1}) = 2$

Individuare una retta con $1 < m < 2$



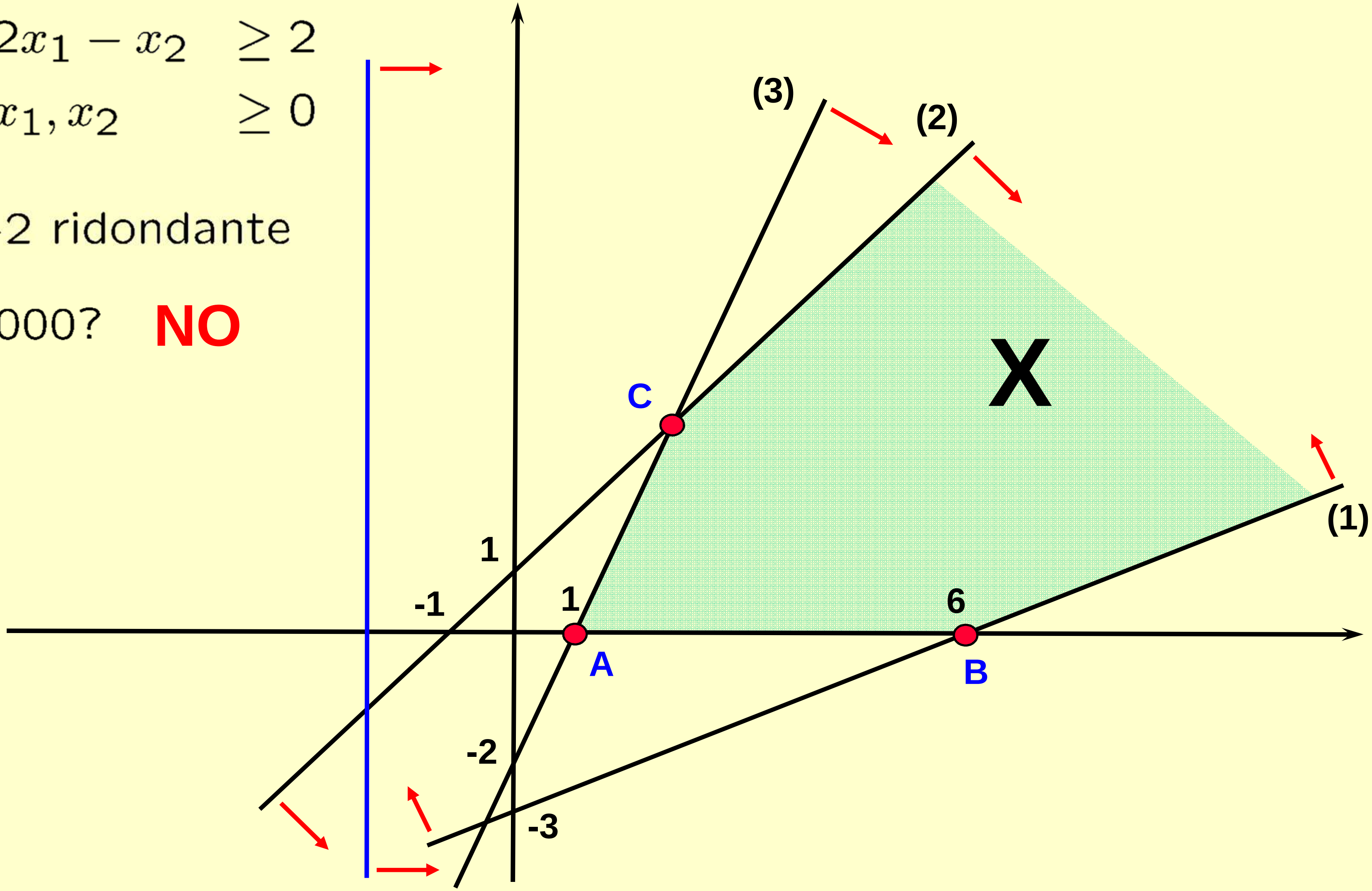
$$\min \; z = 3x_1 + x_2$$

$$
\begin{aligned}
&(1) \quad \frac{1}{2}x_1 - x_2 \leq 3 \\
&(2) \quad -x_1 + x_2 \leq 1 \\
&(3) \quad 2x_1 - x_2 \geq 2 \\
&(4) \quad x_1, x_2 \geq 0
\end{aligned}
$$

e) Aggiungere un vincolo RIDONDANTE

$x_1 \geq -2$ ridondante

$x_1 \geq 1000?$ **NO**

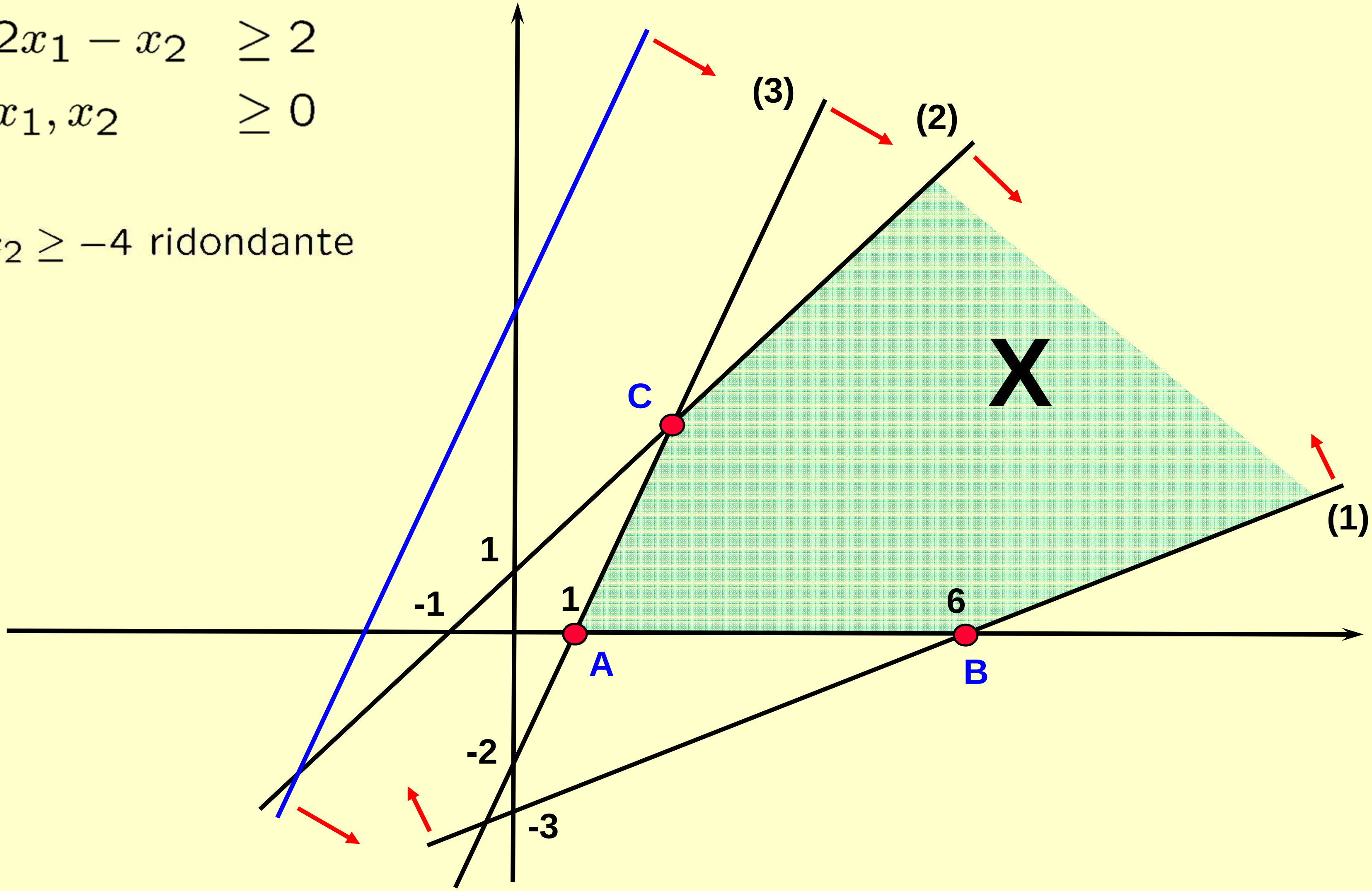


$$\min \; z = 3x_1 + x_2$$

$$\begin{array}{lll} (1) & \frac{1}{2}x_1 - x_2 & \leq 3 \\ (2) & -x_1 + x_2 & \leq 1 \\ (3) & 2x_1 - x_2 & \geq 2 \\ (4) & x_1, x_2 & \geq 0 \end{array}$$

e) Aggiungere un vincolo RIDONDANTE

$2x_1 - x_2 \geq -4$ ridondante

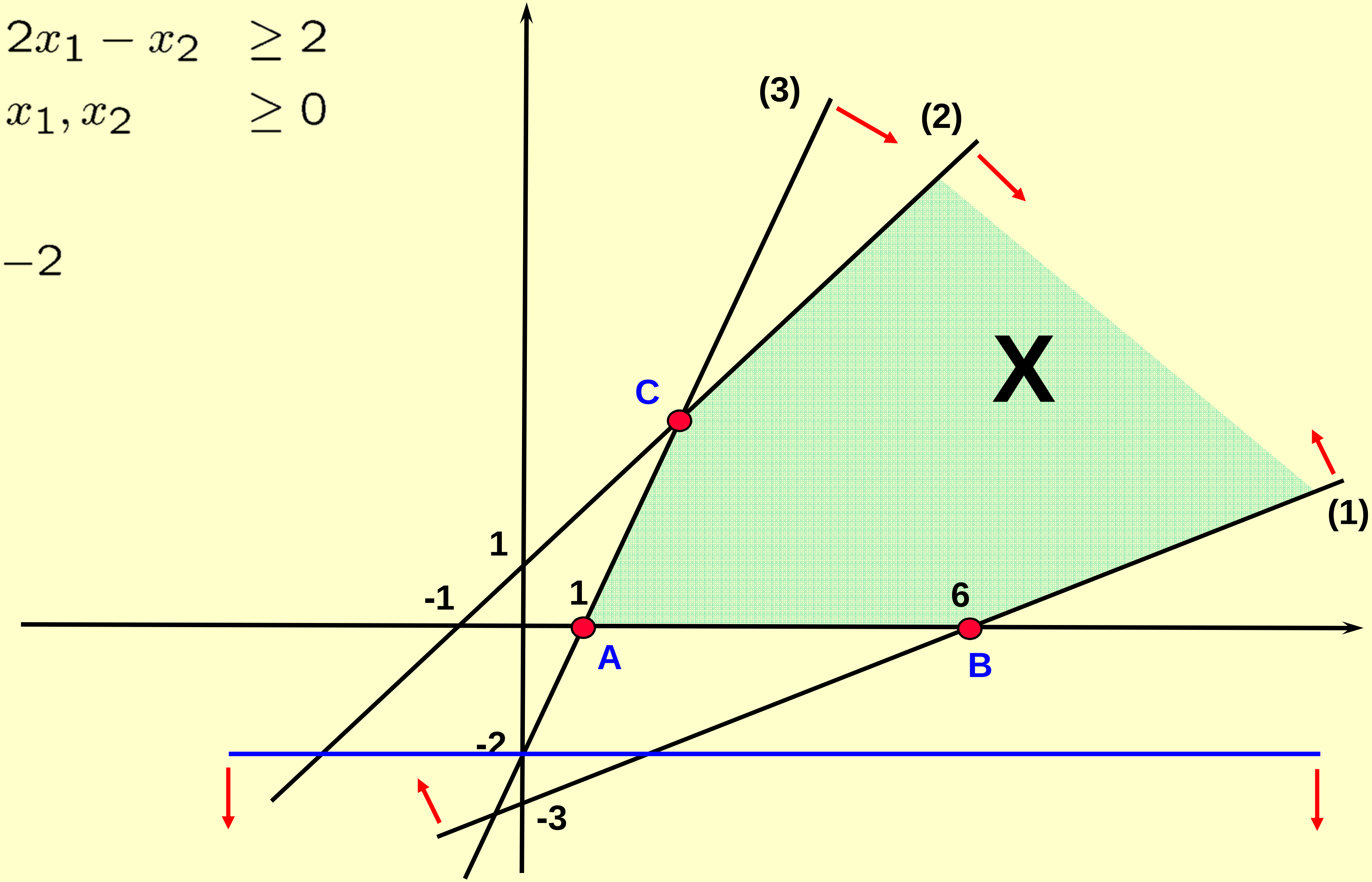


$$\min \ z = 3x_1 + x_2$$

$$\begin{array}{lll} (1) & \frac{1}{2}x_1 - x_2 & \leq 3 \\ (2) & -x_1 + x_2 & \leq 1 \\ (3) & 2x_1 - x_2 & \geq 2 \\ (4) & x_1, x_2 & \geq 0 \end{array}$$

$$x_2 \leq -2$$

f) Aggiungere un vincolo che renda in sistema INAMMISSIBILE

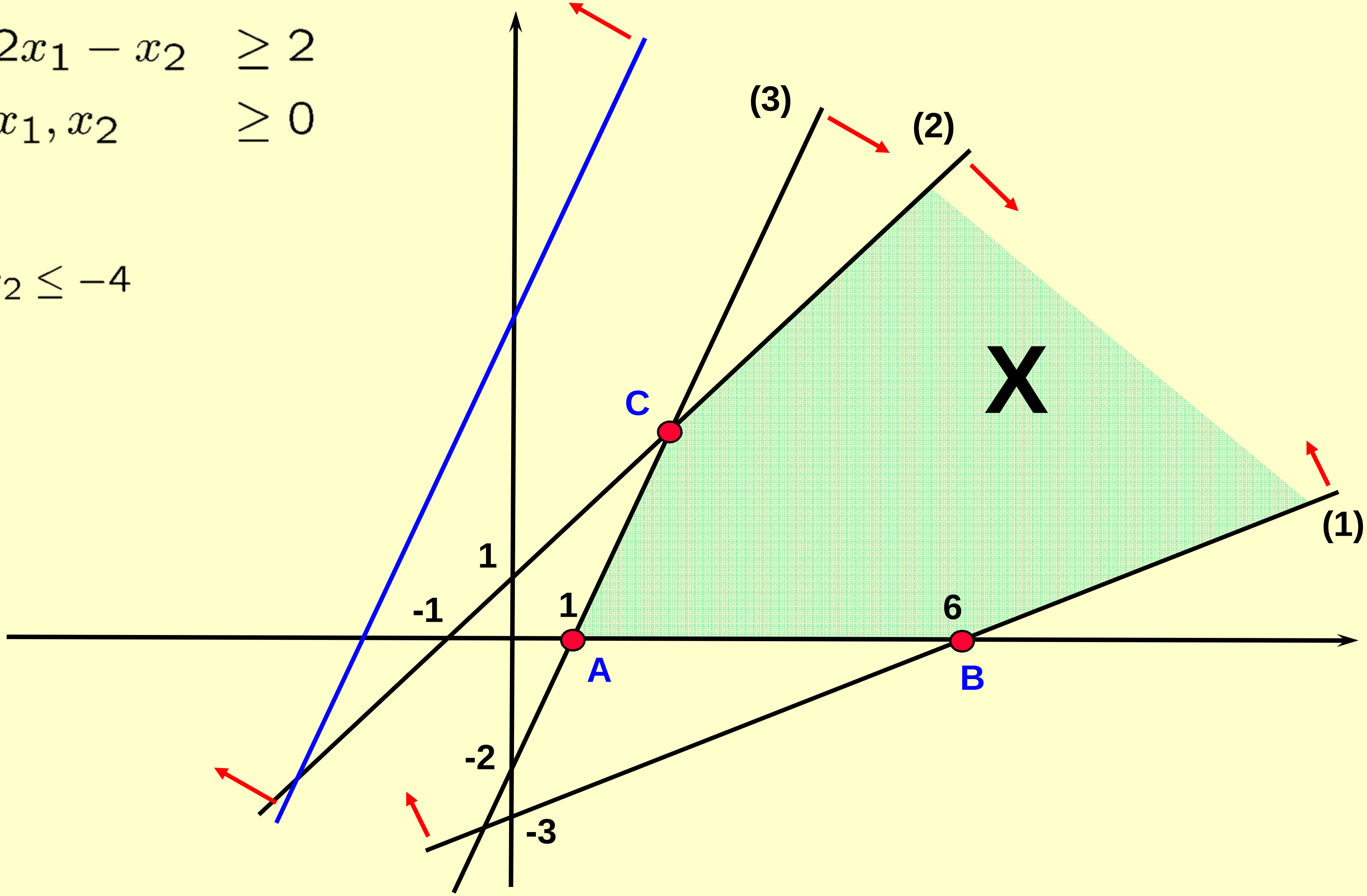


$$\min \ z = 3x_1 + x_2$$

$$\begin{aligned}
&(1) \quad \tfrac{1}{2}x_1 - x_2 \leq 3 \\
&(2) \quad -x_1 + x_2 \leq 1 \\
&(3) \quad 2x_1 - x_2 \geq 2 \\
&(4) \quad x_1, x_2 \geq 0
\end{aligned}$$

f) Aggiungere un vincolo che renda in sistema INAMMISSIBILE

$$2x_1 - x_2 \leq -4$$



$$\min \; z = 3x_1 + x_2$$

$$(1) \quad \frac{1}{2}x_1 - x_2 \leq 3$$

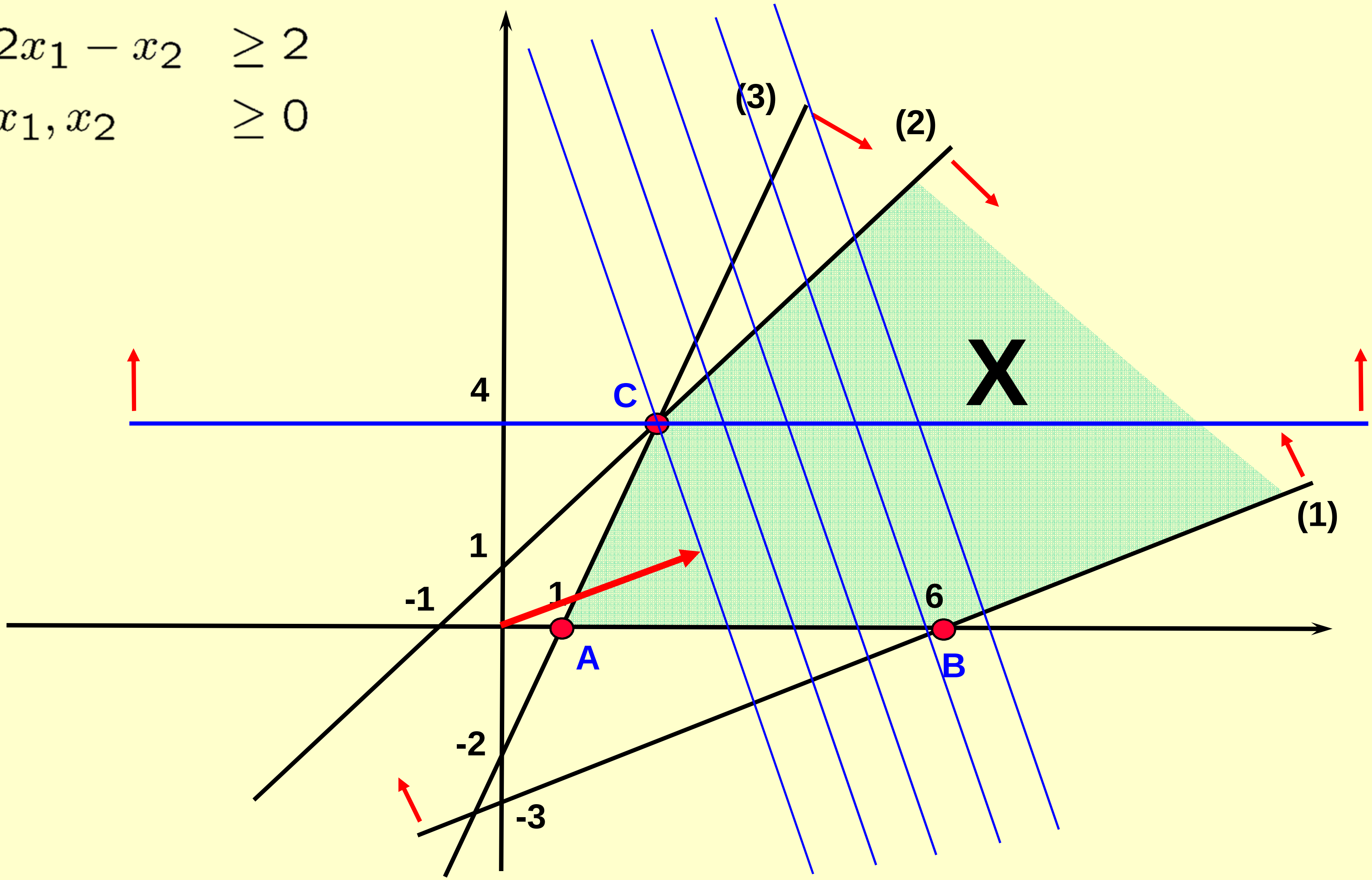
$$(2) \quad -x_1 + x_2 \leq 1$$

$$(3) \quad 2x_1 - x_2 \geq 2$$

$$(4) \quad x_1, x_2 \geq 0$$

g) Aggiungere un vincolo affinchè il punto C diventi punto di ottimo

$x_2 \geq 4$

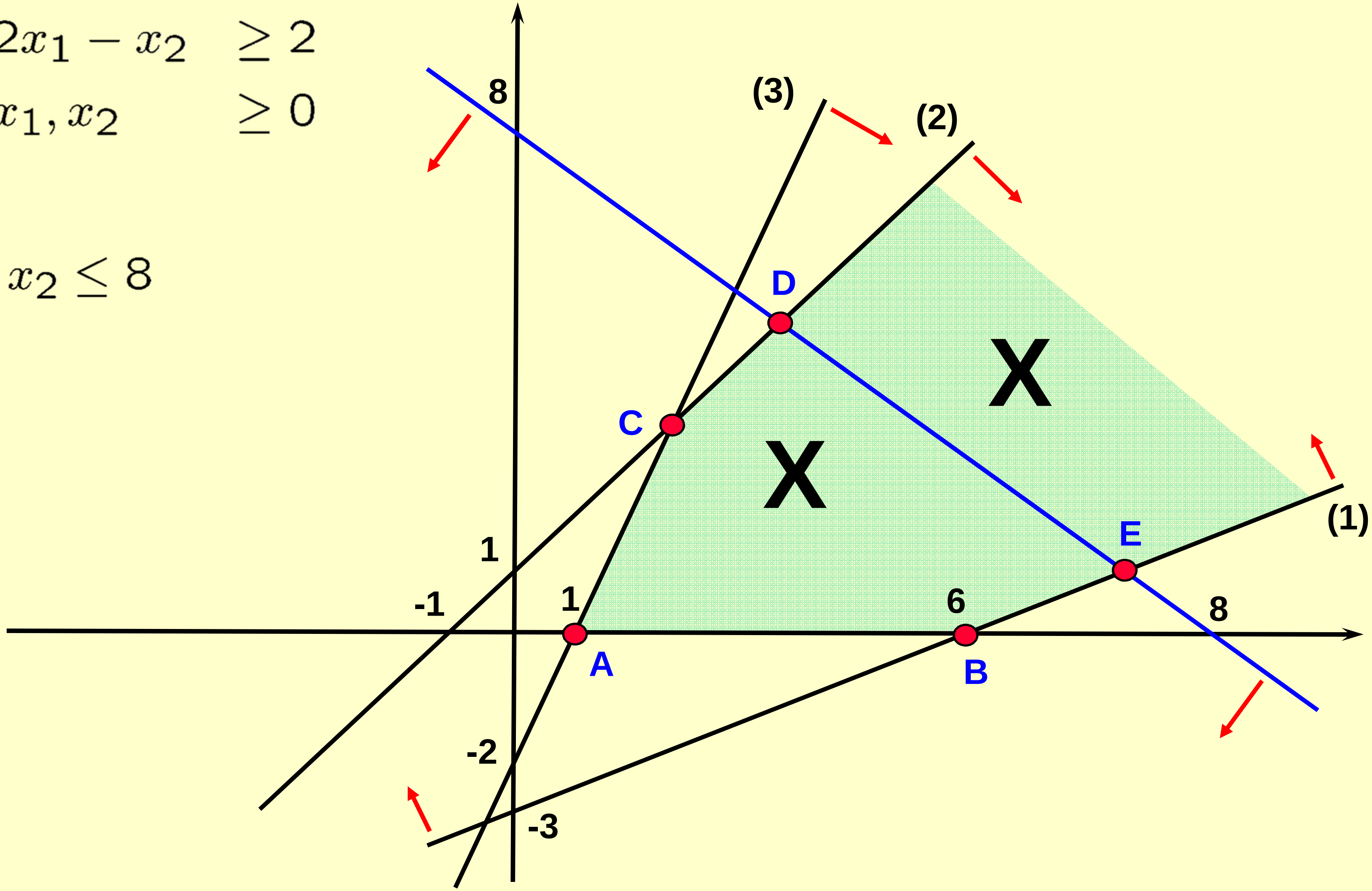


$$\min \ z = 3x_1 + x_2$$

$$\begin{aligned} &(1) \quad \tfrac{1}{2}x_1 - x_2 \leq 3 \\ &(2) \quad -x_1 + x_2 \leq 1 \\ &(3) \quad 2x_1 - x_2 \geq 2 \\ &(4) \quad x_1, x_2 \geq 0 \end{aligned}$$

$$x_1 + x_2 \leq 8$$

f) Aggiungere un vincolo che renda la regione ammissibile un politopo.



$$\min \quad z = 3x_1 + x_2$$

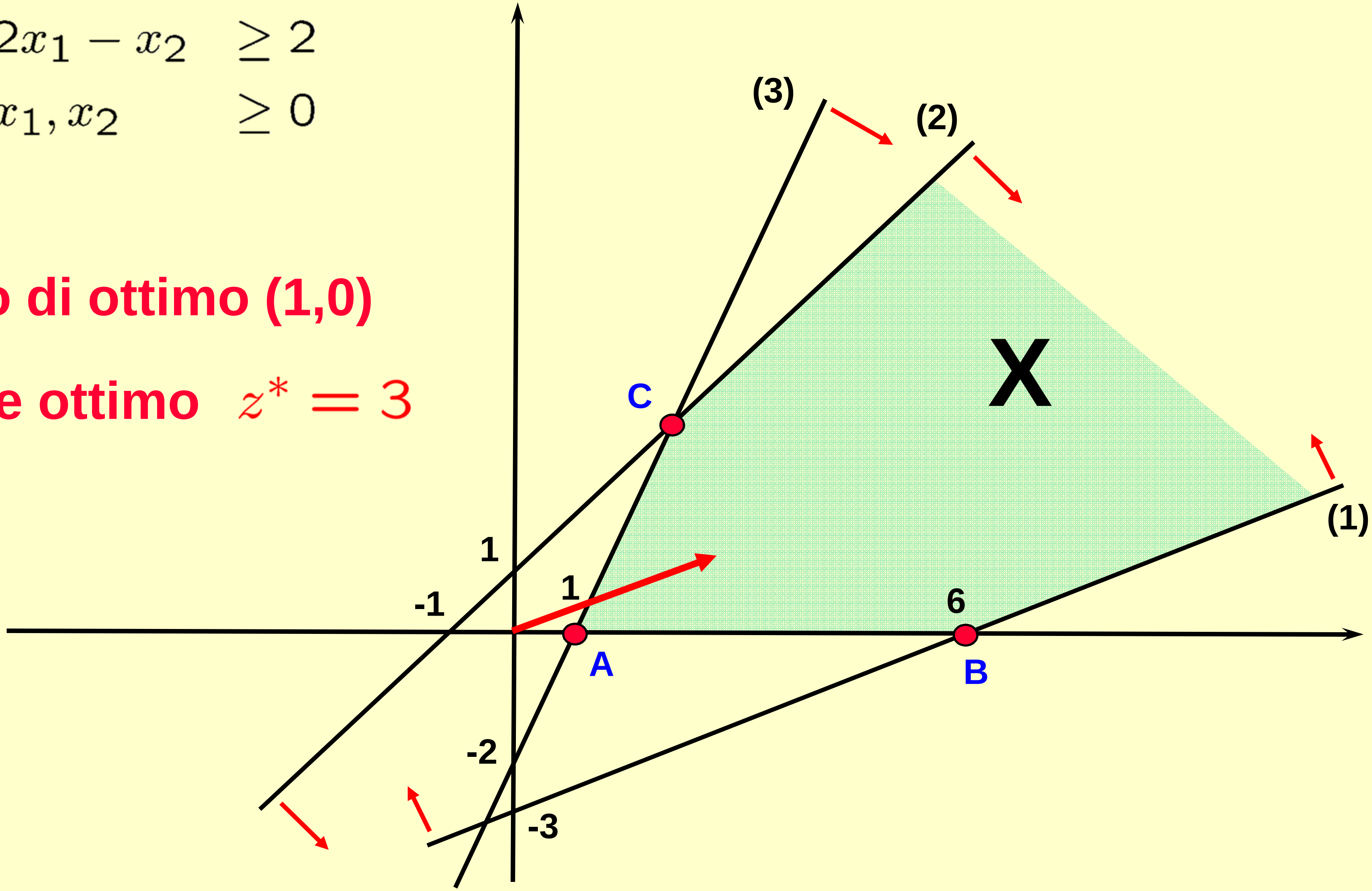
$$(1) \quad \frac{1}{2}x_1 - x_2 \leq 3$$

$$(2) \quad -x_1 + x_2 \leq 1$$

$$(3) \quad 2x_1 - x_2 \geq 2$$

$$(4) \quad x_1, x_2 \geq 0$$

g) Riscrivere il problema applicando il teorema della rappresentazione e risolverlo

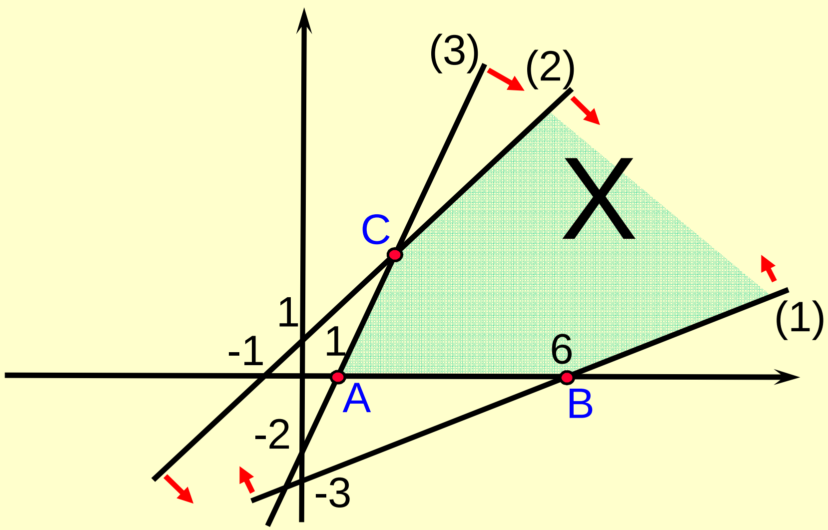


Punto di ottimo (1,0)

Valore ottimo $z^* = 3$

$$min\ z = 3x_1 + x_2$$

$$\begin{array}{lll}(1) & \frac{1}{2}x_1 - x_2 & \leq 3 \\ (2) & -x_1 + x_2 & \leq 1 \\ (3) & 2x_1 - x_2 & \geq 2 \\ (4) & x_1, x_2 & \geq 0\end{array}$$



Calcoliamo punti estremi e le direzioni estreme

A= (1,0)

B= (6,0)

C= ?

$$\begin{cases} -x_1 + x_2 & = 1 \\ 2x_1 - x_2 & = 2 \end{cases} \Rightarrow \begin{cases} -x_1 + x_2 & = 1 \\ x_2 & = 2x_1 - 2 \end{cases}$$

$$\begin{cases} -x_1 + 2x_1 - 2 & = 1 \\ x_2 & = 2x_1 - 2 \end{cases} \Rightarrow \begin{cases} x_1 & = 3 \\ x_2 & = 4 \end{cases}$$

C= (3,4)

$$\min\ z = 3x_1 + x_2$$

$$
\begin{array}{lll}
(1) & \frac{1}{2}x_1 - x_2 & \leq 3 \\
(2) & -x_1 + x_2 & \leq 1 \\
(3) & 2x_1 - x_2 & \geq 2 \\
(4) & x_1, x_2 & \geq 0
\end{array}
$$

A= (1,0) , B= (6,0) , C= (3,4)

$$X = \{ \underline{x} : A\underline{x} \leq \underline{b}, \underline{x} \geq \underline{0} \}$$ (poliedro)

Abbiamo visto che *d* è una a direzione di X se:

$$A\underline{d} \leq 0$$

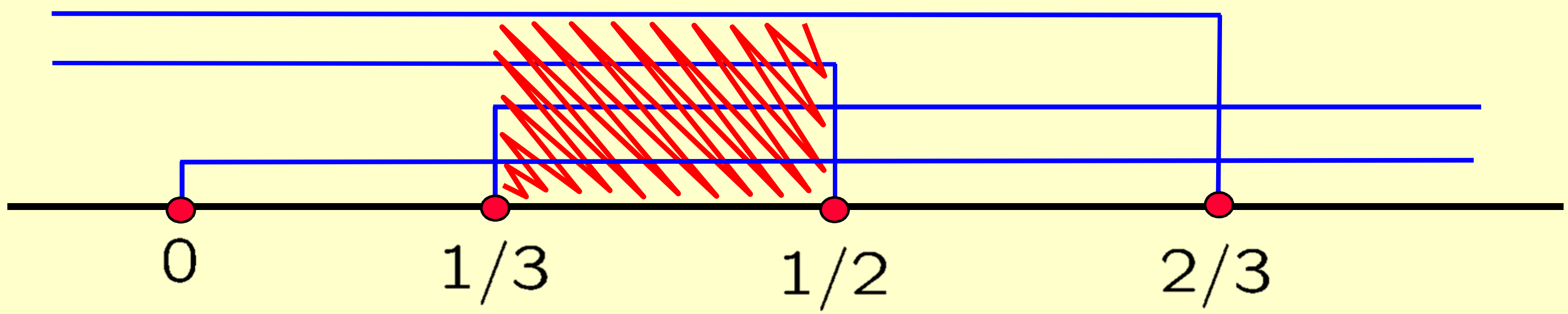
$$\underline{d} \geq 0$$

$$\underline{d} \neq 0$$

$$
\left\{
\begin{array}{ll}
\frac{1}{2}d_1 - d_2 & \leq 0 \\
-d_1 + d_2 & \leq 0 \\
2d_1 - d_2 & \geq 0 \\
d_1 + d_2 & = 1 \\
d_1 \geq 0, d_2 & \geq 0
\end{array}
\right.
\quad \Longrightarrow \quad
\left\{
\begin{array}{ll}
\frac{1}{2}d_1 - d_2 & \leq 0 \\
-d_1 + d_2 & \leq 0 \\
2d_1 - d_2 & \geq 0 \\
d_1 & = 1 - d_2 \\
d_1 \geq 0, d_2 & \geq 0
\end{array}
\right.
$$

$$\begin{cases} \frac{1}{2}d_1 - d_2 & \leq 0 \\ -d_1 + d_2 & \leq 0 \\ 2d_1 - d_2 & \geq 0 \\ d_1 & = 1 - d_2 \\ d_1 \geq 0, d_2 & \geq 0 \end{cases} \Rightarrow \begin{cases} \frac{1}{2} - \frac{1}{2}d_2 - d_2 & \leq 0 \\ -1 + d_2 + d_2 & \leq 0 \\ 2 - 2d_2 - d_2 & \geq 0 \\ d_1 & = 1 - d_2 \\ d_1 \geq 0, d_2 & \geq 0 \end{cases}$$

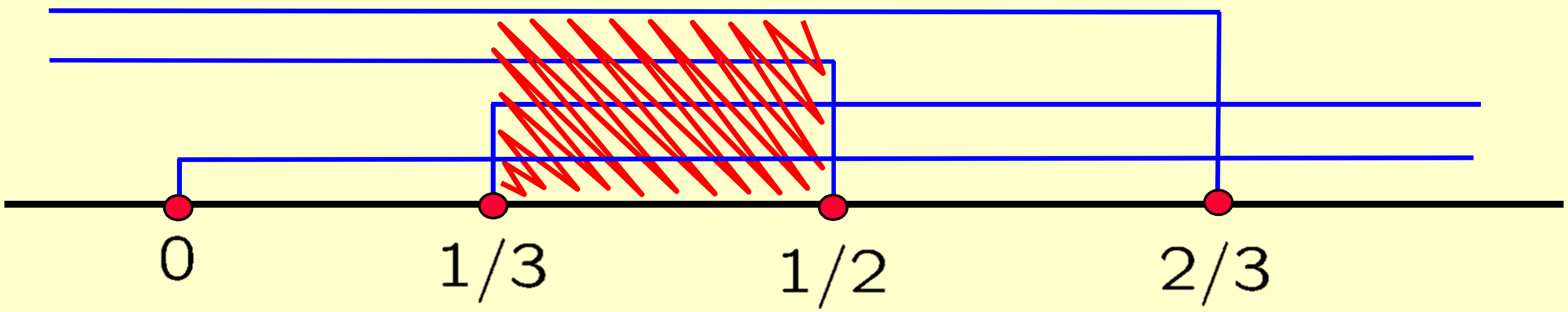
$$\begin{cases} -\frac{3}{2}d_2 & \leq -\frac{1}{2} \\ d_2 & \leq \frac{1}{2} \\ -3d_2 & \geq -2 \\ d_1 & = 1 - d_2 \\ d_1 \geq 0, d_2 & \geq 0 \end{cases} \Rightarrow \begin{cases} d_2 & \geq 1/3 \\ d_2 & \leq 1/2 \\ d_2 & \leq 2/3 \\ d_1 & = 1 - d_2 \\ d_1 \geq 0, d_2 & \geq 0 \end{cases}$$



$$\begin{cases} d_2 & \geq 1/3 \\ d_2 & \leq 1/2 \\ d_2 & \leq 2/3 \\ d_1 & = 1 - d_2 \\ d_1 \geq 0, d_2 & \geq 0 \end{cases}$$

$$d^1 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$d^2 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$



$$min\ z = 3x_1 + x_2$$

$$\begin{array}{lll} (1) & \frac{1}{2}x_1 - x_2 & \leq 3 \\ (2) & -x_1 + x_2 & \leq 1 \\ (3) & 2x_1 - x_2 & \geq 2 \\ (4) & x_1, x_2 & \geq 0 \end{array}$$

$$\min \quad z = \sum_{i=1}^{k} (\underline{c}^T \underline{x}_i)\, \lambda_i + \sum_{j=1}^{t} (\underline{c}^T \underline{d}_j) \mu_j$$

$$\sum_{1=1}^{k} \lambda_i = 1, \quad \lambda_i \geq 0 \quad \mathrm{i} = 1,2,\ldots,\mathrm{k}$$

$$\mu_j \geq 0 \quad \mathrm{j} = 1,2,\ldots,\mathrm{t}$$

A= (1,0)

B= (6,0)

C= (3,4)

$$\min \quad z = \lambda_1 (3\ 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 (3\ 1) \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \lambda_3 (3\ 1) \begin{pmatrix} 3 \\ 4 \end{pmatrix} +$$

$$+ \mu_1 (3\ 1) \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} + \mu_2 (3\ 1) \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$d^1 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$d^2 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$\min \quad z = 3\lambda_1 + 18\lambda_2 + 13\lambda_3 + \frac{7}{3}\mu_1 + 2\mu_2$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

$$\mu_1, \mu_2 \geq 0$$

$$\min\ z = 3x_1 + x_2$$

$$
\begin{aligned}
&(1) \quad \tfrac{1}{2}x_1 - x_2 &\le 3\\
&(2) \quad -x_1 + x_2 &\le 1\\
&(3) \quad 2x_1 - x_2 &\ge 2\\
&(4) \quad x_1, x_2 &\ge 0
\end{aligned}
$$

$$\min \quad z = \sum_{i=1}^{k} (\underline{c}^T \underline{x}_i)\,\lambda_i + \sum_{j=1}^{t} (\underline{c}^T \underline{d}_j)\mu_j$$

$$\sum_{1=1}^{k} \lambda_i = 1, \qquad \lambda_i \ge 0 \quad \mathrm{i} = 1,2,\ldots,\mathrm{k}$$

$$\mu_j \ge 0 \quad \mathrm{j} = 1,2,\ldots,\mathrm{t}$$

A= (1,0)

B= (6,0)

C= (3,4)

$$d^1 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$d^2 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Valore ottimo $z^* = 3$

minimo

$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$

$\underline{c}^T \underline{d}^1 > 0 \quad \underline{c}^T \underline{d}^2 > 0$

$$\min \quad z = 3\lambda_1 + 18\lambda_2 + 13\lambda_3 + \frac{7}{3}\mu_1 + 2\mu_2$$

$u_1 = 0 \quad u_2 = 0$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\lambda_1, \lambda_2, \lambda_3 \ge 0$$

$$\mu_1, \mu_2 \ge 0$$

$$\min \ z = 3x_1 + x_2$$

$$(1) \quad \frac{1}{2}x_1 - x_2 \leq 3$$

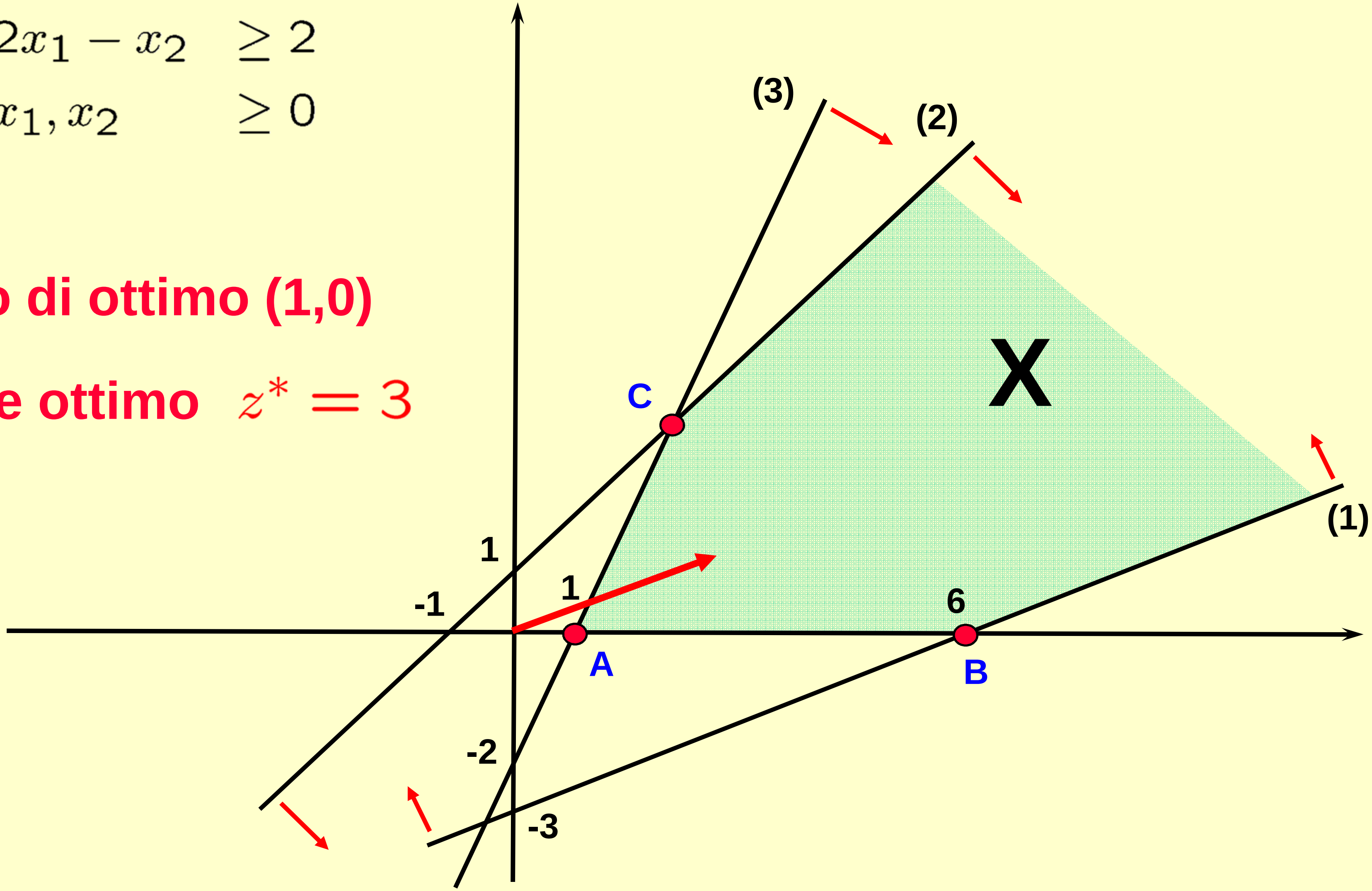
$$(2) \quad -x_1 + x_2 \leq 1$$

$$(3) \quad 2x_1 - x_2 \geq 2$$

$$(4) \quad x_1, x_2 \geq 0$$

Punto di ottimo (1,0)

Valore ottimo $z^* = 3$



Dato il seguente problema di programmazione lineare:

$$\max \; z = x_1 + x_2$$

$$\begin{array}{lll} (1) & 3x_1 + 5x_2 & \geq 1 \\ (2) & x_2 & \leq 3 \\ (3) & x_1 + x_2 & \leq 5 \\ (4) & x_1, x_2 & \geq 0 \end{array}$$

a) Risolvere per esso tutti i punti dell'esercizio precedente