Prova del 13/09/2011

1. $\lim_{x \to \pi/2} \sqrt{1-\sin x} = 0$

2. $f(x) = \sqrt[3]{(x-1)(x-2)^2}$

3. $f(x) = \begin{cases} e^{(x-\alpha)} & x>0 \\ \sqrt{1-\sin\beta x} & x \le 0 \end{cases}$

4. $A = \left\{ \frac{(-1)^n}{1+e^n} : n \in \mathbb{N} \right\}$

5. $\lim_{x \to 0} \frac{1-x^2-e^{-x}-\sin x}{\sin^3 x + \sqrt{1+\tan^2 x} - 1}$

6. $\int e^{-2x} \ln(1+e^{-x})\,dx$

7. $\sum_{n=1}^{\infty} \left(\ln\left(\tan \frac{1}{n}\right) - \ln \frac{1}{n} \right)$

8. $f(x,y) = \frac{x}{y} + \frac{y}{x}$

1. $\lim\limits_{x \to \pi/2} \sqrt{1-\sin x} = 0$

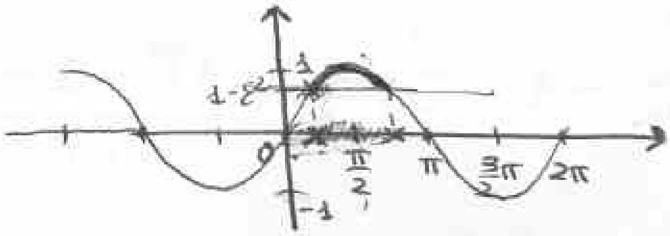
$\forall \varepsilon > 0 \ \exists^{?} I(\pi/2) : \sqrt{1-\sin x} < \varepsilon \quad \forall x \in I(\pi/2)$

D: $1 - \sin x \geq 0$

$\sin x - 1 \leq 0$

$\sin x \leq 1$

$\forall x \in \mathbb{R}$



$\sqrt{1-\sin x} < \varepsilon \iff 1 - \sin x < \varepsilon^2 \iff -\sin x < \varepsilon^2 - 1$

$\iff \sin x > 1 - \varepsilon^2 \iff$

$\frac{\pi}{2} - \arcsin(1-\varepsilon^2) < x < \frac{\pi}{2} + \arcsin(1-\varepsilon^2)$

$I\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2} - \arcsin(1-\varepsilon^2) ; \frac{\pi}{2} + \arcsin(1-\varepsilon^2) \right)$

2. $\begin{cases} f'_-(0) = f'_+(0) \\ e^{-\alpha} = 1 \end{cases}$ $\quad \begin{cases} \left(\frac{-\beta \cos \beta x}{2\sqrt{1-\sin \beta x}} \right)_{x=0} = \left(e^{x-\alpha}\right)_{x=0} \\ e^{-\alpha} = e^{0} \end{cases}$

$\begin{cases} \frac{-\beta}{2} = e^{-\alpha} \\ \alpha = 0 \end{cases}$ $\quad \begin{cases} -\beta = 2e^{-\alpha} \\ \alpha = 0 \end{cases}$ $\quad \begin{cases} \beta = -2 \\ \alpha = 0 \end{cases}$

4. La succ. è limitata: $-1 < \frac{(-1)^n}{1+e^n} < 1$

infatti:

$$\begin{cases} \frac{(-1)^n}{1+e^n} + 1 > 0 \iff \frac{(-1)^n + 1 + e^n}{1+e^n} > 0 \\ \frac{(-1)^n}{1+e^n} - 1 < 0 \iff \frac{(-1)^n - 1 - e^n}{1+e^n} < 0 \end{cases} \quad \forall n \in \mathbb{N}$$

$\inf A = -1 \notin A$ perciò non è min.

$\sup A = 1 \notin A$ perciò non è max.

5. $$\lim_{x\to 0} \frac{1 - x^2 - e^{-x} - \sin x}{\sin^3 x + \sqrt{1+\tan^2 x} - 1} =$$

$$\lim_{x\to 0} \frac{1 - x^2 - e^{-x} - \sin x}{\sin^3 x + \left|\frac{1}{\cos x}\right| - 1} =$$

$\sin^3 x = \underline{\sin x}(1-\cos^2 x)$

$$\lim_{x\to 0} \frac{\cancel{1} - x^2 - \cancel{1} + \cancel{x} - \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) - \cancel{x} + \frac{x^3}{6} + o(x^4)}{\left(x - \frac{x^3}{6} + o(x^4)\right)\left(x^2 - \frac{x^4}{3} + o(x^5)\right) + \cancel{1} + \frac{x^2}{2} + o(x^3) - \cancel{1}}$$

$$\lim_{x\to 0} \frac{-\frac{3}{2}x^2 + \frac{x^3}{3} + o(x^3)}{\underline{x^3} - \frac{x^5}{3} + o(x^6) - \frac{x^5}{6} + \frac{x^7}{18} + o(x^8) + o(x^6) + o(x^8) + o(x^9) + \cancel{1} + \frac{x^2}{2} + o(x^3) - \cancel{1}}$$

$$\lim_{x\to 0} \frac{-\frac{3}{2}x^2 + \frac{x^3}{3} + o(x^3)}{\frac{x^2}{2} + x^3 + o(x^3)} = \lim_{x\to 0} \frac{\cancel{x^2}\left(-\frac{3}{2} + \frac{x}{3} + \frac{o(x^3)}{x^2}\right)}{\cancel{x^2}\left(1/2 + x + \frac{o(x^3)}{x^2}\right)}$$

$$= -3.$$

$$\int e^{-2x} \cdot \ln(1+e^{-x})\,dx = -\frac{1}{2}\int \underbrace{-2e^{-2x}}_{D(e^{-2x})} \ln(1+e^{-x})\,dx$$

$$= -\frac{1}{2}\left[e^{-2x} \cdot \ln(1+e^{-x}) - \int e^{-2x} \cdot \frac{1}{1+e^{-x}} \cdot e^{-x} \cdot (-1)\,dx \right] =$$

$$= -\frac{1}{2}\left[e^{-2x} \ln(1+e^{-x}) + \int \frac{e^{-3x}}{1+e^{-x}}\,dx^{*} \right] =$$

$$^{*}\ e^{-x} = t \rightarrow \log e^{-x} = \log t \rightarrow -x = \log t \rightarrow x = -\log t$$
$$\rightarrow dx = -\frac{1}{t}\,dt$$

$$\int \frac{e^{-3x}}{1+e^{-x}}\,dx = \int \frac{t^{\cancel{3}\,2}}{1+t} \cdot -\frac{1}{\cancel{t}}\,dt = -\int \frac{t^2}{1+t}\,dt$$

$$= -\int \frac{t^2-1}{t+1}\,dt - \int \frac{1}{t+1}\,dt = -\int (t-1)\,dt - \int \frac{dt}{t+1}$$

$$= -\frac{t^2}{2} + t - \log|t+1| + c,\ c \in \mathbb{R} \underset{t=e^{-x}}{=}$$

$$= -\frac{e^{-2x}}{2} + e^{-x} - \log|e^{-x}+1| + c,\ c \in \mathbb{R}$$

7.

$$\sum_{n=1}^{+\infty} \left(\ln\left(\tan\frac{1}{n}\right) - \ln\frac{1}{n} \right) = \sum_{n=1}^{+\infty} \ln\left(\frac{\tan\frac{1}{n}}{\frac{1}{n}}\right)$$

$$\lim_{n \to +\infty} \ln\left(\frac{\tan\frac{1}{n}}{\frac{1}{n}}\right) = 0 \Rightarrow$$ La serie o conv. o div.

Siccome $x \leq \tan x \Rightarrow \frac{1}{n} \leq \tan\frac{1}{n} \Rightarrow$

$\log\frac{1}{n} \leq \log\left(\tan\frac{1}{n}\right) \Rightarrow$ la serie è a termini non neg.

8. $f(x,y) = \frac{x}{y} + \frac{y}{x}$

$D: \mathbb{R}^2 - \{(0,0), (0,y), (x,0)\}$ con $x, y \in \mathbb{R}$

$f_x = \frac{1}{y} - yx^{-2}$ $\qquad f_y = -xy^{-2} + \frac{1}{x}$

$f_{xx} = +2yx^{-3}$ $\qquad f_{yy} = +2xy^{-3}$

$f_{xy} = -y^{-2} - x^{-2}$

$$\begin{cases} \frac{1}{y} - \frac{y}{x^2} = 0 \\ \frac{1}{x} - \frac{x}{y^2} = 0 \end{cases} \qquad \begin{cases} \frac{x^2 - y^2}{\cancel{x^2 y}} = 0 \\ \frac{y^2 - x^2}{\cancel{xy^2}} = 0 \end{cases} \qquad \begin{cases} x^2 = y^2 \rightarrow x = \pm\sqrt{y} \\ x^2 = y^2 \end{cases}$$

$A(\sqrt{y}, y)$

$B(-\sqrt{y}, y)$

$$Hf(x,y) = \begin{vmatrix} \frac{2y}{x^3} & -\frac{1}{y^2} - \frac{1}{x^2} \\ -\frac{1}{y^2} - \frac{1}{x^2} & \frac{2x}{y^3} \end{vmatrix} = \frac{4xy}{x^3y^3} - \left(-\frac{1}{y^2} - \frac{1}{x^2}\right)^2$$

$$= \frac{4}{x^2y^2} - \frac{1}{y^4} - \frac{1}{x^4} - \frac{2}{x^2y^2} = \frac{2}{x^3y^2} - \frac{1}{y^4} - \frac{1}{x^4} =$$

$$-\left(\frac{1}{x^4}+\frac{1}{y^4}-\frac{2}{x^2y^2}\right) = -\left(\frac{1}{x^2}-\frac{1}{y^2}\right)^2 < 0 \quad \forall (x,y) \in D$$

dominio

Siccome l'Hessiano è sempre negativo non ci sono punti nè di min nè di max.

$n, m \in \mathbb{N}$

$n < m \Longrightarrow a_n < a_m$

$n < m \Longrightarrow \dfrac{3^n}{1+3^n} < \dfrac{3^m}{1+3^m}$

$(1+2008-1)/x+2008 = x+2008$