CHAPTER 10 (corrisponde al cap. 9 italiano)

Error Detection and Correction

Solutions to Review Questions and Exercises

Review Questions

1. In a ***single bit error*** only one bit of a data unit is corrupted; in a **burst error** more than one bit is corrupted (not necessarily contiguous).
2. ***Redundancy*** is a technique of adding extra bits to each data unit to determine the accuracy of transmission.
3. In ***forward error correction***, the receiver tries to correct the corrupted codeword; in ***error detection by retransmission***, the corrupted message is discarded (the sender needs to retransmit the message).
4. A ***linear block code*** is a block code in which the exclusive-or of any two codewords results in another codeword. A ***cyclic code*** is a linear block code in which the rotation of any codeword results in another codeword.
5. The ***Hamming distance*** between two words (of the same size) is the number of differences between the corresponding bits. The Hamming distance can easily be found if we apply the XOR operation on the two words and count the number of 1s in the result. The ***minimum Hamming distance*** is the smallest Hamming distance between all possible pairs in a set of words.
6. The ***single parity check*** uses one redundant bit for the whole data unit. In a ***two-dimensional parity check***, original data bits are organized in a table of rows and columns. The parity bit is then calculated for each column and each row.
7.
   a. The only relationship between the size of the codeword and dataword is the one based on the definition: **n = k + r**., where ***n*** is the size of the codeword, ***k*** is the size of the dataword, and ***r*** is the size of the remainder.
   b. The ***remainder*** is always ***one bit smaller*** than the ***divisor***.
   c. The ***degree*** of the generator polynomial is ***one less than*** the size of the ***divisor***. For example, the CRC-32 generator (with the polynomial of degree 32) uses a 33-bit divisor.

d. The ***degree*** of the generator polynomial is the ***same as*** the size of the remainder (length of checkbits). For example, CRC-32 (with the polynomial of degree 32) creates a remainder of 32 bits.

8. ***One's complement arithmetic*** is used to add data items in checksum calculation. In this arithmetic, when a number needs more than *n* bits, the extra bits are wrapped and added to the number. In this arithmetic, the complement of a number is made by inverting all bits.

9. ***At least three types of error*** cannot be detected by the current checksum calculation. First, if two data items are swapped during transmission, the sum and the checksum values will not change. Second, if the value of one data item is increased (intentionally or maliciously) and the value of another one is decreased (intentionally or maliciously) the same amount, the sum and the checksum cannot detect these changes. Third, if one or more data items is changed in such a way that the change is a multiple of $2^{16} - 1$, the sum or the checksum cannot detect the changes.

10. ***The value of a checksum can be all 0s*** (in binary). This happens when the value of the sum (after wrapping) becomes all 1s (in binary). ***It is almost impossible for the value of a checksum to be all 1s***. For this to happen, the value of the sum (after wrapping) must be all 0s which means all data units must be 0s.

Exercises

11. We can say that **(vulnerable bits) = (data rate) × (burst duration)**

a.	vulnerable bits	$= (1{,}500) \times (2 \times 10^{-3})$	= **3 bits**
b.	vulnerable bits	$= (12 \times 10^{3}) \times (2 \times 10^{-3})$	= **24 bits**
c.	vulnerable bits	$= (100 \times 10^{3}) \times (2 \times 10^{-3})$	= **200 bits**
d.	vulnerable bits	$= (100 \times 10^{6}) \times (2 \times 10^{-3})$	= **200,000 bits**

Comment: The last example shows how a noise of small duration can affect so many bits if the data rate is high.

12.

a.	(10001)	⊕	(10000)	= **00001**
b.	(10001)	⊕	(10001)	= **00000**
c.	(11100)	⊕	(00000)	= **11100**
d.	(10011)	⊕	(11111)	= **01100**

Comment: The above shows three properties of the exclusive-or operation. First, the result of XORing two equal patterns is an all-zero pattern (part b). Second, the result of XORing of any pattern with an all-zero pattern is the original non-zero pattern (part c). Third, the result of XORing of any pattern with an all-one pattern is the complement of the original non-one pattern.

13. The codeword for dataword **10** is **101**. This codeword will be changed to **010** if a 3-bit burst error occurs. This pattern is not one of the valid codewords, so the receiver detects the error and discards the received pattern.

14. The codeword for dataword **10** is **10101**. This codeword will be changed to **01001** if a 3-bit burst error occurs. This pattern is not one of the valid codewords, so the receiver discards the received pattern.

15.
    a. d (10000, 00000) = **1**
    b. d (10101, 10000) = **2**
    c. d (1111, 1111) = **0**
    d. d (000, 000) = **0**

    Comment: Part c and d show that the distance between a codeword and itself is 0.

16.
    a. For error detection $\rightarrow$ $d_{min} = s + 1 = 2 + 1 =$ **3**
    b. For error correction $\rightarrow$ $d_{min} = 2t + 1 = 2 \times 2 + 1 =$ **5**
    c.
    For error section $\rightarrow$ $d_{min} = s + 1 = 3 + 1 =$ **4**
    For error correction $\rightarrow$ $d_{min} = 2t + 1 = 2 \times 2 + 1 =$ **5**
    Therefore d_{min} should be **5**.
    d.
    For error detection $\rightarrow$ $d_{min} = s + 1 = 6 + 1 =$ **7**
    For error correction $\rightarrow$ $d_{min} = 2t + 1 = 2 \times 2 + 1 =$ **5**
    Therefore d_{min} should be **7**.

17.
    a. **01**
    b. **error**
    c. **00**
    d. **error**

18. We show that the exclusive-or of the second and the third code word

$$(01011) \oplus (10111) = \mathbf{11100}$$

is not in the code. The code is not linear.

19. We check five random cases. All are in the code.

I.	(1st)	$\oplus$	(2nd)	=	(2nd)
II.	(2nd)	$\oplus$	(3th)	=	(4th)
III.	(3rd)	$\oplus$	(4th)	=	(2nd)
IV.	(4th)	$\oplus$	(5th)	=	(8th)
V.	(5th)	$\oplus$	(6th)	=	(2nd)

20. We show the dataword, the codeword, the corrupted codeword, and the interpretation of the receiver for each case:
    a. Dataword: **0100** $\rightarrow$ Codeword: **0100011** $\rightarrow$ Corrupted: **0010011**
    This pattern is not in the table. $\rightarrow$ **Correctly discarded**.
    b. Dataword: **0111** $\rightarrow$ Codeword: **0111001** $\rightarrow$ Corrupted: **1111000**
    This pattern is not in the table. $\rightarrow$ **Correctly discarded.**

c. Dataword: **1111** → Codeword: **1111111** → Corrupted: **0101110**
This pattern is in the table. → **Erroneously accepted as 0101.**

d. Dataword: **0000** → Codeword: **0000000** → Corrupted: **1101000**
This pattern is in the table. → **Erroneously accepted as 1101.**

Comment: The above result does not mean that the code can never detect three errors. The last two cases show that it may happen that three errors remain undetected.

21. We show the dataword, codeword, the corrupted codeword, the syndrome, and the interpretation of each case:

a. Dataword: 0100 → Codeword: 0100011 → Corrupted: **1100011** → $s_2s_1s_0$ = 110
Change b_3 (Table 10.5) → Corrected codeword: **0100011** → dataword: 0100
The dataword is correctly found.

b. Dataword: 0111 → Codeword: 0111001 → Corrupted: **0011001** → $s_2s_1s_0$ = 011
Change b_2 (Table 10.5) → Corrected codeword: **0111001**→ dataword: 0111
The dataword is correctly found.

c. Dataword: 1111 → Codeword: 1111111 → Corrupted: **0111110** → $s_2s_1s_0$ = 111
Change b_1 (Table 10.5) → Corrected codeword: **0101110**→ dataword: 0101
The dataword is found, but it is **incorrect**. C(7,4) cannot correct two errors.

d. Dataword: 0000 → Codeword: 0000000 → Corrupted: **1100001** → $s_2s_1s_0$ = 100
Change q_2 (Table 10.5) → Corrected codeword: **1100101**→ dataword: 1100
The dataword is found, but it is **incorrect**. C(7,4) cannot correct three errors.

22.

a. If we rotate **0101100** one bit, the result is **0010110**, which is in the code. If we rotate **0101100** two bits, the result is **0001011**, which is in the code. And so on.

b. The XORing of the two codewords (0010110) ⊕ (1111111) = 1101001, which is in the code.

23. We need to find $k = 2^m - 1 - m \geq 11$. We use *trial and error* to find the right answer:

a. Let m = 1 k = $2^m - 1 - m = 2^1 - 1 - 1 = 0$ (not acceptable)

b. Let m = 2 k = $2^m - 1 - m = 2^2 - 1 - 2 = 1$ (not acceptable)

c. Let m = 3 k = $2^m - 1 - m = 2^3 - 1 - 3 = 4$ (not acceptable)

d. Let m = 4 k = $2^m - 1 - m = 2^4 - 1 - 4 = 11$ (acceptable)

Comment: The code is **C(15, 11)** with **d_{min} = 3**.

24.

a. $(x^3 + x^2 + x + 1) + (x^4 + x^2 + x + 1) = x^4 + x^3$

b. $(x^3 + x^2 + x + 1) - (x^4 + x^2 + x + 1) = x^4 + x^3$

c. $(x^3 + x^2) \times (x^4 + x^2 + x + 1) = x^7 + x^6 + x^5 + x^2$

d. $(x^3 + x^2 + x + 1) / (x^2 + 1) = x + 1$ (remainder is 0)

25.

a. 101110 → $x^5 + x^3 + x^2 + x$

b. 101110 → 101110**000** (Three 0s are added to the right)

c. $x^3 \times (x^5 + x^3 + x^2 + x) = x^8 + x^6 + x^5 + x^4$

d. 101110 $\rightarrow$ 10 (The four rightmost bits are deleted)

e. $x^{-4} \times (x^5 + x^3 + x^2 + x) = x$ (Note that negative powers are deleted)

26. To detect single bit errors, a CRC generator must have at least two terms and the coefficient of x^0 must be nonzero.

a. $x^3 + x + 1 \rightarrow$ It meets both critic.

b. $x^4 + x^2 \rightarrow$ It meets the first criteria, but not the second.

c. $1 \rightarrow$ It meets the second criteria, but not the first.

d. $x^2 + 1 \rightarrow$ It meets both criteria.

27. CRC-8 generator is $x^8 + x^2 + x + 1$.

a. It has more than one term and the coefficient of x^0 is 1. It can detect a single-bit error.

b. The polynomial is of degree 8, which means that the number of checkbits (remainder) r = 8. It will detect all burst errors of size 8 or less.

c. Burst errors of size 9 are detected most of the time, but they slip by with probability $(1/2)^{r-1}$ or $(1/2)^{8-1} \approx 0.008$. This means **8 out of 1000** burst errors of size 9 are left undetected.

d. Burst errors of size 15 are detected most of the time, but they slip by with probability $(1/2)^r$ or $(1/2)^8 \approx 0.004$. This means **4 out of 1000** burst errors of size 15 are left undetected.

28. This generator is $x^{32} + x^{26} + x^{23} + x^{22} + x^{16} + x^{12} + x^{11} + x^{10} + x^8 + x^7 + x^5 + x^4 + x^2 + x + 1$.

a. It has more than one term and the coefficient of x^0 is 1. It detects all single-bit error.

b. The polynomial is of degree 32, which means that the number of checkbits (remainder) r = 32. It will detect all burst errors of size 32 or less.

c. Burst errors of size 33 are detected most of the time, but they are slip by with probability $(1/2)^{r-1}$ or $(1/2)^{32-1} \approx 465 \times 10^{-12}$. This means **465 out of 10^{12}** burst errors of size 33 are left undetected.

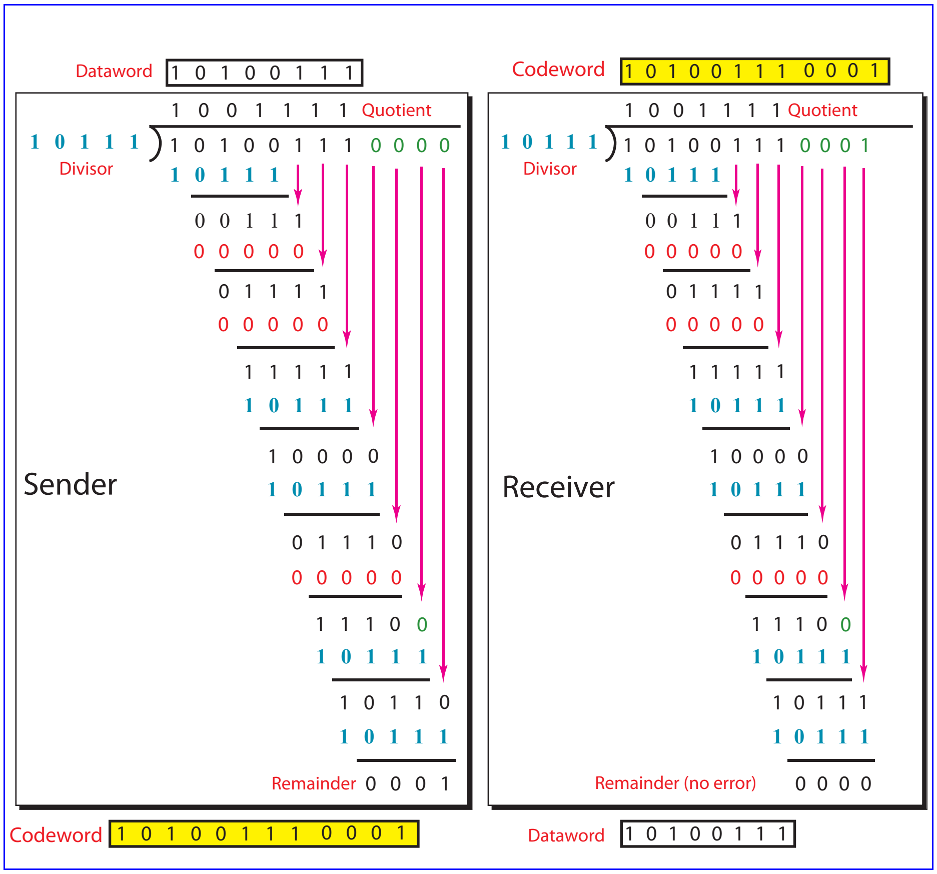
d. Burst errors of size 55 are detected most of the time, but they are slipped with probability $(1/2)^r$ or $(1/2)^{32} \approx 233 \times 10^{-12}$. This means **233 out of 10^{12}** burst errors of size 55 are left undetected.

29. We need to add all bits modulo-2 (XORing). However, it is simpler to count the number of 1s and make them even by adding a 0 or a 1. We have shown the parity bit in the codeword in color and separate for emphasis.

	Dataword		Number of 1s		Parity	Codeword
a.	1001011	$\rightarrow$	4 (even)	$\rightarrow$	**0**	**0** 1001011
b.	0001100	$\rightarrow$	2 (even)	$\rightarrow$	**0**	**0** 0001100
c.	1000000	$\rightarrow$	1 (odd)	$\rightarrow$	**1**	**1** 1000000
d.	1110111	$\rightarrow$	6 (even)	$\rightarrow$	**0**	**0** 1110111

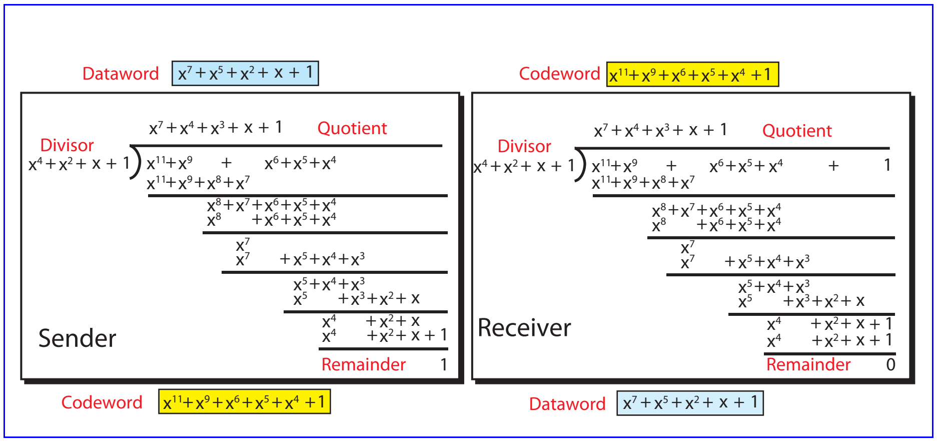
30. Figure 10.1 shows the calculation of the checksum both at the sender and receiver site using binary division.

Figure 10.1 *Solution to Exercise 30*



31. Figure 10.2 shows the generation of the codeword at the sender and the checking of the received codeword at the receiver using polynomial division.

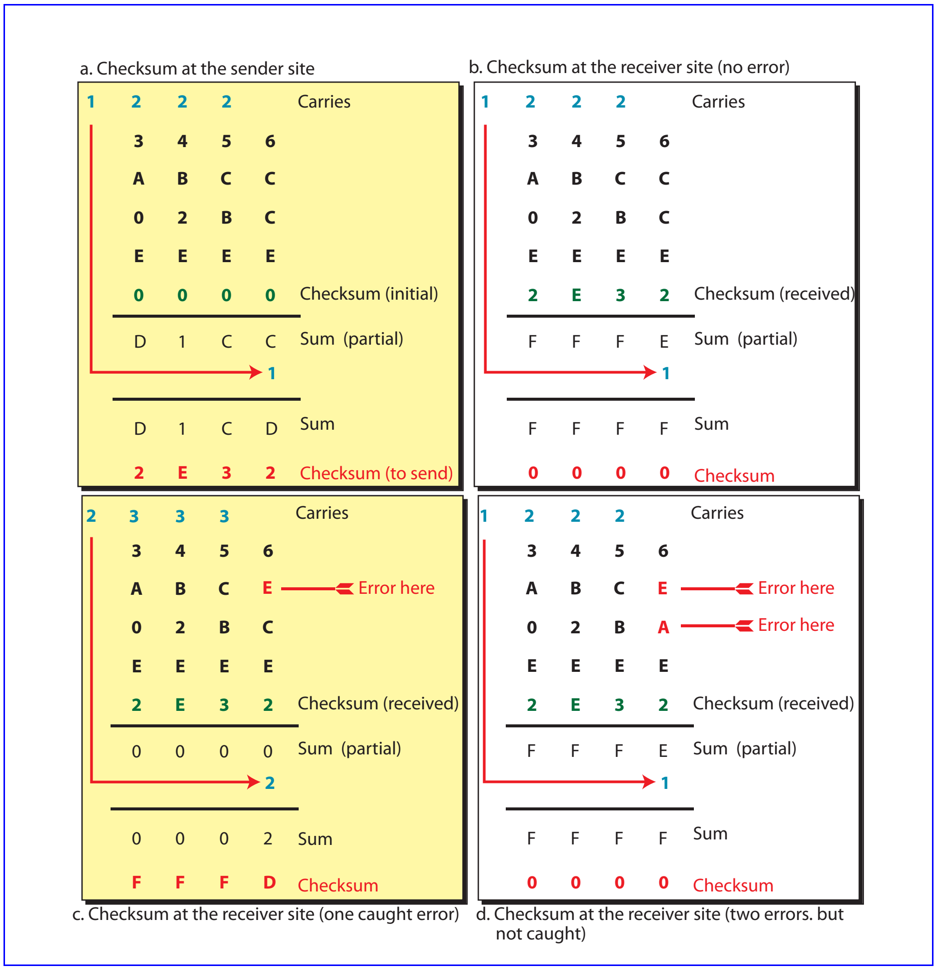
Figure 10.2 *Solution to Exercise 31*



Dataword $x^7+x^5+x^2+x+1$
Codeword $x^{11}+x^9+x^6+x^5+x^4+1$
Divisor x^4+x^2+x+1
Quotient $x^7+x^4+x^3+x+1$
Sender
Receiver
Remainder 1
Remainder 0
Codeword $x^{11}+x^9+x^6+x^5+x^4+1$
Dataword $x^7+x^5+x^2+x+1$

32. Figure 10.3 shows the four situations.

Figure 10.3 *Solution to Exercise 32*



a. In part a, we calculate the checksum to be sent (0x2E32)

b. In part b, there is no error in transition. The receiver recalculates the checksum to be all 0x0000. The receiver correctly assumes that there is no error.

c. In part c, there is one single error in transition. The receiver calculates the checksum to be 0FFFD. The receiver correctly assumes that there is some error and discards the packet.

d. In part d, there are two errors that cancel the effect of each other. The receiver calculates the checksum to be 0x0000. The receiver erroneously assumes that there is no error and accepts the packet. This is an example that shows that the checksum may slip in finding some types of errors.

33. Figure 10.4 shows the checksum to send (0x0000). This example shows that the checksum ***can be all 0s***. ***It can be all 1s only if all data items are all 0, which means no data at all***.

Figure 10.4 *Solution to Exercise 33*

```
4   5   6   7
B   A   9   8
0   0   0   0   Checksum (initial)
----------------
F   F   F   F   Sum
0   0   0   0   Checksum (to send)
```